DECLINE IN DISSOLVED OXYGEN DUE TO INCREASING TEMPERATURE AND ALGAL BLOOMS: MATHEMATICAL MODEL

Shreya¹ and Preety Kalra²
Department of Mathematics/School of Chemical Engineering and Physical Sciences/Lovely Professional University, Phagwara, Punjab, India. 
E-mail: ¹shreyatangri25@gmail.com , ²kalra.preety@gmail.com

Abstract
The dissolved oxygen is one of the primary resources required for the survival of the species in an aquatic ecosystem. However, in present times, the dissolved oxygen in water bodies is decreasing due to the increasing carbon dioxide level in the environment as well as the increasing global warming. The effects of these two factors taken together are studied in a mathematical model consisting of variables as average surface temperature, carbon dioxide concentration, density of algal bloom in water and concentration of dissolved oxygen in water. Analytically, stability conditions have been derived and numerically, threshold levels for carbon input have been derived

Keywords : Carbon Dioxide, Global Warming, Algal Blooms, Dissolved oxygen, Stability, Model

Introduction
In recent times, the human activities are leading to a steep rise in the pollution and the carbon dioxide level in the environment. The marine environment is exposed to toxins, toxic chemicals, industrial wastes, etc. by their direct input in water. The presence of pollutants in water results in habitat degradation of aquatic species (He and Wang, 2009). Increased fossil fuel burning, industrial pollutants etc. are some of the factors which lead to rise of carbon emissions in the environment (Shaffer et al., 2009). The increased atmospheric carbon dioxide level contributes to global warming (Solomon et al., 2009). According to Sweetman et al., the increase in atmospheric carbon emissions would result in an increase of 1 degree in ocean temperature and a reduction in dissolved oxygen. This will further impact aquatic species’ availability of food and alter their growth and survival rates (Sweetman et al., 2017). Climate change also has a direct influence on an ecosystem’s structure and functions. The way materials and energy flow in an environment is heavily influenced by climate change (Sarmiento et al., 2010). Under global warming, the species’ metabolic rates are also changed (Danovaro et al., 2001). Studies have also revealed alteration in the relationship for aquatic species because of changes in temperature of water (Yvon-Durocher et al., 2010). Increased temperatures also lead to melting of ice sheets which proves adverse to the survival of human population (Shukla et al., 2017). Additional carbon dioxide level can further result in weak acidification of the water body when occurring in systems with low buffering ability (Barker and Ridgwell, 2012, Raven et al., 2005, Stets et al., 2017). Poor acidification in the marine environment has the potential to harm reproduction abilities in macrophytes in water and cause alterations in phytoplankton communities (Hasler et al., 2018). The rising acidity, contamination and rising water temperatures lead to an increased growth of algal blooms in the water bodies (Chapra et al., 2017, Michalak et al., 2013, Winter et al., 2011). Climate change is also found to have a significant impact on the frequency and abundance of algal blooms affecting the growth and habitat of the algal blooms. The growing algal blooms have negative affect on ecosystem functioning (Glibert et al., 2014). They also disturb the aquatic biodiversity and food chains (Riebesell et al., 2018). It is seen that, due to pollutants and increased algal growth present in the water bodies, not only the species are affected but the resource i.e. dissolved oxygen is also decreased. The oxygen levels in water fall with rise in algal growth as the algal blooms use up the oxygen in their decomposition process (Bhatia and Jain, 2016, O’Boyle et al., 2016). Ocean warming caused by global climate change results in deoxygenation with negative consequences for ocean productivity and marine habitat. Ocean models predict declines of 1 to 7 percent over the next century in the global ocean oxygen stocks, with declines extending into the future for a thousand years or more (Keeling et al., 2010). Thus, it is found that global warming is responsible for expansion of hypoxic zones in aquatic bodies as it leads to deoxygenation with reduced global oxygen content and negative effects on ocean productivity and marine habitat (Carlensen et al., 2014). The condition of hypoxia due to global warming may also lead to death of various aquatic species dependent on oxygen for their survival (Joos et al., 2003). Increased acidity and global warming also lead to the loss of coral reefs’ biodiversity, which poses a threat to their dependent populations (Miller et al., 2009, Wolff et al., 2018). Therefore, due to the growing acidity and hypoxic conditions, aquatic species face difficulty in their growth and survival (Jansson et al., 2015).

The inflating menace of water contamination has been elucidated by various researchers (Kaur et al., 2017, Patel et al., 2020, Rashid et al., 2016, Shukla et al., 2019, Singh et al., 2018, Singh et al., 2015). Certain researchers have also proposed certain studies to handle the problem of water pollution (Bansal and Geetha, 2018, Bhandari et al., 2018, Bhatia et al., 2017, Dhanjal et al., 2018, Garg et al., 2018, Jain et al., 2019, Kannari and Dutta, 2018, Kanjilal et al., 2014, Kaur et al., 2018, Kalra and Kamboj, 2019, Kushwaha and Gupta, 2018, Parihar et al., 2015, Rahul et al., 2018, Sharma et al., 2016, Sharma et al., 2018, Vise et al., 2018). Mathematical modeling has been used a powerful aid in many studies to study biological processes (Kalra and Kumar, 2018, Kumar and Kumar, 2019, Sahoo and Patra, 2020, Yadav and Priyanka, 2019, Yadav, 2019, Kalra and Kumar, 2018). Various mathematical and
experimental studies are available which support the decrease of dissolved oxygen due to global warming (Sekerci and Petrovskii, 2018). Mukherjee et al. developed a mathematical model for changes in inorganic carbon due to respiration, photosynthesis and deposition of calcium carbonate in aquatic systems (Mukherjee et al., 2002). Also, certain mathematical studies study the effect of pollutants and acidification on aquatic ecosystem individually (Chakraborty et al., 2017, Shukla et al., 2008). But, a combined study which studies the effects of global warming, carbon emissions and algal bloom growth together in one mathematical model is not yet available.

In our study, we have formulated a mathematical model of non-linear differential equations to study the impact of global warming, increased carbon emissions and increased algal bloom growth on the level of dissolved oxygen in water. Threshold levels for maximum amount of carbon input that the aquatic ecosystem can handle has been calculated, above which the carbon dioxide input increase can lead to condition of hypoxia having dreadful effects on aquatic populations. Also, threshold level of oxygen input has also been calculated required to sustain the aquatic life under increased carbon dioxide concentration and global warming. It has been found that under the effects of global warming and carbon dioxide increase in the atmosphere, the dissolved oxygen decreases which can have very harmful effects on the aquatic ecosystem.

Mathematical Model

It is assumed in the proposed mathematical model that due to increase of carbon dioxide concentration in the environment, the atmospheric temperature is increasing which is leading to warming of waters. Also, the water acidity is increasing due to the increasing carbon concentration. These factors are leading to increase in growth of algal blooms in the water bodies. The increasing algal blooms and water temperature further cause decline in the concentration of dissolved oxygen in water. Under these assumptions, let \( \tau(t) \) be the average surface temperature, \( R(t) \) be the concentration of atmospheric carbon dioxide, \( A_g(t) \) be the density of algal blooms and \( O(t) \) be the dissolved oxygen concentration in water.

The model is formulated as given below:

\[
\frac{d\tau}{dt} = \Phi(R - R_0) - \beta(A_g - A_{g0})
\]
\[
\frac{dR}{dt} = Q - nR - \frac{\beta R A_g}{1 + \beta R}
\]
\[
\frac{dT}{dt} = \frac{R T}{1 + \beta R} - \beta A_g
\]
\[
\frac{dO}{dt} = I - dA_g - nO - d_2 O
\]

with initial conditions

\[ A_g(0) \geq A_{g0}, R(0) \geq R_0, T(0) \geq T_0, 0, O(0) \geq 0. \]

The model parameters are as given below:

\( \Phi \) is the growth rate coefficient corresponding to average surface temperature, \( \beta \) is the coefficient of depletion of average surface temperature, \( R_0 \) is the level of carbon dioxide in absence of pollution and human activities, \( T_0 \) is the average surface temperature in absence of rising carbon dioxide levels, \( Q \) is the rate of increase of carbon dioxide due to human activities, \( n \) is the natural depletion rate of carbon concentration. \( I \) gives the growth rate of algal blooms due to increasing carbon concentration in water and rising water acidity and \( \alpha \) represents the proportionality constant. \( \beta \) is the natural death rate of algal blooms. \( J \) is the input rate of oxygen in water. \( \delta \) gives the depletion rate of dissolved oxygen due to decomposition process of algal blooms in water. \( n \) is the natural depletion rate of oxygen and \( d_2 \) gives the rate of decrease of dissolved oxygen concentration due to low solubility of oxygen in water due to global warming.

Dynamical Behaviour

In order to carry out the analysis of the model given by equations (1) - (4), the dynamical behaviour of the mathematical model shall be studied in this section.

1. Boundedness and Positive Invariance

In order to show the boundedness of the solutions of the model given by equations (1)-(4), the following lemma shall be proved.

**Lemma 3.1** All the solutions of the system of equations given by (1)-(4) lie in the region

\[ \mathcal{W}_1 = \left\{ (\tau, R, A_g, O) \in \mathbb{R}^4 : \tau \leq \tau_0, 0 \leq R, 0 \leq A_g \leq A_{g0}, 0 \leq O \leq W_{O1} \right\} \]

as \( t \to \infty \). for all initial values \( \tau(0), R(0), A_g(0), O(0) \) as given for the model consisting of equations (1)-(4) where,

\[ \tau_0 = \frac{\Phi R_0 + \delta T_0}{\delta}, R_0 = \frac{I}{n}, A_{g0} = \frac{W_{O1} + \delta}{\delta}, b_{11} = \alpha n \tau_0(\alpha \beta, n) \]

**Proof:** From equation (2) we get,

\[ \frac{dR}{dt} \leq Q - nR \]

then by usual comparison theorem, as \( t \to \infty \),

\[ \limsup_{t \to \infty} R(t) \leq \frac{Q}{n} = R_N \]

From equation (1) we get,

\[ \limsup_{t \to \infty} \tau(t) \leq \frac{\Phi R_0 + \delta T_0}{\delta} = \tau_N \]

From equations (2)-(4),

\[ \frac{d(R + A_g + O)}{dt} \leq Q + I - nR - \beta A_g - nO \]

then by comparison theorem we get as \( t \to \infty \),

\[ \limsup_{t \to \infty} (R + A_g + O, t) \leq \frac{Q + I}{b_{11}} = W_{O1} \]

where \( b_{11} = \min(\alpha \beta, n) \).

From equation (4),

\[ \frac{dO}{dt} \leq I - nO \]

By comparison theorem we get as \( t \to \infty \),

\[ \limsup_{t \to \infty} O(t) \leq \frac{I}{n} = O_M \]

**Theorem 3.2:** The box \( \mathcal{W}_1 \) in the space \((\tau, R, A_g, O)\) is compact and positive invariant.
Proof: Consider a box \( \mathcal{V} \) in space \((\tau_1, R, A_A, O)\). One vertex of the box is considered to be at origin and the other vertex at \(V = (\tau', R', A'_A, O')\). Also, consider \( \tau' > \tau, R > R_A, A_A > A_A' \) and \( O' > O \). For calculating the angle of flow that is made with each face of \( \mathcal{V} \), not lying in coordinate planes, let \( p_1, p_2, p_3 \) and \( p_4 \) be the outward unit normal vectors to planes \( G_2; \tau = \tau', G_2: R = R', G_2: A_A = A_A' \) and \( G_4: O = O' \).

We get from equation (4),

\[
\frac{d\tau}{dt}\big|_{G_2} \leq 1 - n\tau' - d_\tau\tau'
\]

Since \( O' \geq \frac{1}{n} \),

\[
\frac{d\tau}{dt}\big|_{G_4} \leq 0
\]

\[
\text{hence } \frac{d\tau}{dt}\big|_{G_4} \leq 0.
\]

Similarly, \( \frac{d\tau}{dt}\big|_{G_1} \leq 0, \frac{d\tau}{dt}\big|_{G_2} \leq 0 \),

\[
\text{Hence, } \mathcal{V} \text{ is the region of attraction for model given by equations (1)-(4).}
\]

2. Equilibrium points

In this sub-section, the feasible equilibria of the system given by equations (1) - (4) shall be calculated.

1. Boundary equilibrium \( E_1(t, 0, 0, 0, 0) \), where \( R = 0, A_A = 0 \) i.e. the carbon dioxide emissions and the algal population density is taken as zero.

\[
\tau = \frac{\delta R - \delta R_0}{\delta}, \tau > 0; \text{where } \delta c_0 - \phi R_0 > 0
\]

\[
\beta = \frac{1}{n + d_\tau}
\]

\[
\overrightarrow{O'} \geq 0 \text{ if } n + d_\tau < 0.
\]

2. Interior equilibrium \( E_2(t, R', A_A', O') \), where,

\[
R' = \frac{\beta}{R - R_0}
\]

\[
R' > 0 \text{ if } k > a\beta
\]

\[
\tau' = \frac{\beta R - \beta R_0 + \beta R_0}{R - R_0}
\]

\[
\tau' > 0 \text{ as } R' \geq R_0
\]

\[
A_A' = \frac{\delta c_0 - \delta c}{\beta R - \beta R_0 + \beta R_0}
\]

\[
A_A' > 0 \text{ if } A_A' > A_A
\]

\[
O' = \frac{1}{(A_A' + n + d_\tau)}
\]

As \( \tau' > 0 \), hence \( O' > 0 \).

3. Local Stability

In this sub-section we shall carry out the local stability analysis for the model given by equations (1) - (4).

1. For the variational matrix associated with the boundary equilibrium \( E_1 \), the characteristic equation is given by

\[
\left[ \begin{array}{cc}
\frac{1}{\delta} - \lambda & -\delta - \lambda \\
-\delta - \lambda & -\beta - \lambda
\end{array} \right] = 0
\]

(14)

\[
\lambda_1 = -\frac{1}{\delta}, \lambda_2 = -\delta, \lambda_3 = -\alpha, \lambda_4 = -\beta
\]

The nature of roots of the equation (14) shows that the equilibrium point \( E_2 \) is asymptotically stable.

2. For the variational matrix associated with the interior equilibrium \( E_2 \), the characteristic equation is given by

\[
(\frac{1}{\delta} - \lambda)(-\delta - \lambda)(-\alpha - \lambda)(-\beta - \lambda) = 0
\]

where

\[
Z_1 = -\alpha - \frac{hA_A^2}{1 + a\tau'} - \frac{hA_A^2}{1 + a\tau'} Z_2 = \frac{hR}{1 + a\tau'}, Z_2 = \frac{hA_A^2}{1 + a\tau'} - \frac{hA_A^2}{1 + a\tau'} Z_2 = 0
\]

(16)

(17)

Using Routh’s criteria, the equilibrium point \( E_2 \) is stable only if the following is satisfied,

\[
\alpha + \frac{hA_A^2}{1 + a\tau'} > \frac{hA_A^2}{1 + a\tau'}
\]

(18)

\[
\frac{1}{1 + a\tau'} < 1
\]

(19)

4. Global Stability

In this sub-section, for the model given by equations (1) - (4), we shall study the global stability behaviour about the interior equilibrium \( E_2 \).

Theorem 4.1 The following inequalities should hold for the model given by the system of equations (1)-(4) to be globally stable.

\[
\frac{\delta}{(1 + a\tau')^2} > \frac{hA_A^2}{1 + a\tau'}
\]

(20)

\[
\text{Proof: Consider a positive definite function,}
\]

\[
\delta \left( \frac{\alpha + \frac{hA_A^2}{1 + a\tau'}}{(1 + a\tau')^2} \right) > \frac{hA_A^2}{1 + a\tau'}
\]

Differentiating w.r.t. \( t \) we get,

\[
\delta \frac{\partial}{\partial \tau} \left( \frac{\alpha + \frac{hA_A^2}{1 + a\tau'}}{(1 + a\tau')^2} \right) + \frac{1}{2} \left( \frac{\alpha + \frac{hA_A^2}{1 + a\tau'}}{(1 + a\tau')^2} \right) = 0
\]

(20)

where,

\[
b_{11} = \beta, b_{22} = \alpha + \frac{hA_A^2}{1 + a\tau'}
\]

\[
b_{22} = \beta + \frac{hR}{1 + a\tau'}, b_{24} = \delta A_A, b_{32} = -\delta
\]

\[
b_{24} = \delta A_A + n + d_\tau, b_{26} = -\delta
\]

Using Sylvester’s criteria we get,

\[
b_{11} b_{22} > b_{12}^2, b_{22} b_{32} > b_{23}^2, b_{22} b_{44} > b_{24}^2, b_{11} b_{44} > b_{14}^2
\]
Hence, the system of equations (1)-(4) shall be globally stable if the following are satisfied,

\[
\begin{align*}
\sigma \left( \frac{\beta + \frac{\ln b}{(1+aR)(1+aR')}}{\beta + \frac{\ln b}{(1-aR)(1+aR')}} \right) & > \phi^2, \\
\left( \frac{\ln b}{1+aR} \right)^2 & > \left( \frac{\ln b}{1-aR} \right)^2, \\
\sigma \left( \frac{\beta + \frac{\ln b}{(1+aR)(1+aR')}}{\beta + \frac{\ln b}{(1-aR)(1+aR')}} \right) & > \left( \frac{\ln b}{1+aR} \right)^2, \\
\sigma (\frac{\ln b}{1-aR}) (d + n + d_1 r) & > (\frac{\ln b}{1+aR}) (d_1 r)^2.
\end{align*}
\]  

(21) \hspace{1cm} (22) \hspace{1cm} (23) \hspace{1cm} (24)

**Numerical Example**

For the equations given in (1) - (4), we shall consider the values of parameters as given below:

\[
\begin{align*}
\phi &= 0.03, \beta = 0.3, R_0 = 5.5, \tau_0 = 13.5, \alpha = 0.04, \\
\beta &= 0.4, a = 0.7, \beta = 0.55, l = 23.44, \\
d &= 0.6, n = 0.3, Q = 6.358, d_2 = 0.001.
\end{align*}
\]

With these values of parameters, the following value of variables at interior equilibrium point \( E_2 (r^*, R^*, A^*, U^*) \) are obtained:

\[
\begin{align*}
 r^* &= 16.6157, \quad R^* = 38.8667, \quad A^*_2 = 8.8983, \\
U^* &= 6.8193.
\end{align*}
\]

For the above mentioned set of values, the conditions of stability given by equations (18) - (19), (21) - (24) are satisfied. Therefore, the interior equilibrium \( E_2 \) is stable. The same is supported by figure (1). Also, the dissolved oxygen equilibrium value is similar to the threshold level of dissolved oxygen concentration in fresh water bodies (Nürnberg, 2002).

**Conclusion**

From the stability analysis of the model given by equations (1) - (4), it is concluded that the existence, boundedness and stability conditions at interior equilibrium \( E_2 \) given by equations (9), (12), (18) - (19), (21) - (24) are satisfied. \( E_2 \) is found to be locally stable as shown in figure (1). Also, the interior equilibrium point is globally stable as shown in figure (2). The stability analysis shows that dissolved oxygen concentration decreases with rise in global warming as shown in figure (3). This is also supported by equation (13). Also, with the increase in carbon dioxide concentration, the dissolved oxygen level decreases as supported by equation (13) and shown in figure (4). It is also observed that if the value of rate of increase of carbon dioxide increases more than \( Q = 6.694 \), value of dissolved oxygen drops below 2.00. Thus, a condition of hypoxia arises which can be detrimental for the aquatic population. Also, at value of carbon dioxide increase rate \( Q = 6.83833 \), the dissolved oxygen level decreases to zero which can lead to fish kills. Hence, we get a threshold level for rate of increase of carbon, above which the survival of species in an aquatic ecosystem is not possible. It is also shown that \( Q = 6.83833 \), if the value of dissolved oxygen input is maintained above \( I = 24.115 \), the dissolved oxygen level again starts to increase. This is shown in figure (5). Thus, for the survival of aquatic species under the carbon dioxide increase rate \( Q \) greater than 6.83833, the dissolved oxygen input in water has to be maintained above \( I = 24.115 \). Thus, it is concluded from our study that the rising global warming and carbon dioxide levels in atmosphere greatly decrease the dissolved oxygen level in water, thus, harming the aquatic ecosystem.
Fig. 4: Decrease in dissolved oxygen with increasing carbon dioxide

Fig. 5: Graph for threshold value of dissolved oxygen

References


